David Hilbert

In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.

preface to *Geometry and the Imagination* (1932)

Karni and Magda

Karni: This one is definitely bigger than 90 degrees. I think it's more like 110 degrees, or 120 degrees.

Magda: Yeah. Actually, I think it's bigger than that. Maybe 120 degrees or 130 degrees. Let's check it and see.

Karni: This one is sixty degrees, I think. Do you agree? (Picking up the green rhombus and pointing to the sixty-degree angle.)

Magda: Yeah, that's what I got, too.

Karni: OK. Let's see how many sixties fit in there.

(Students composed the angle with two sixty-degree angles.)

Magda: Sixty plus 60 is 120. It is 120 degrees!

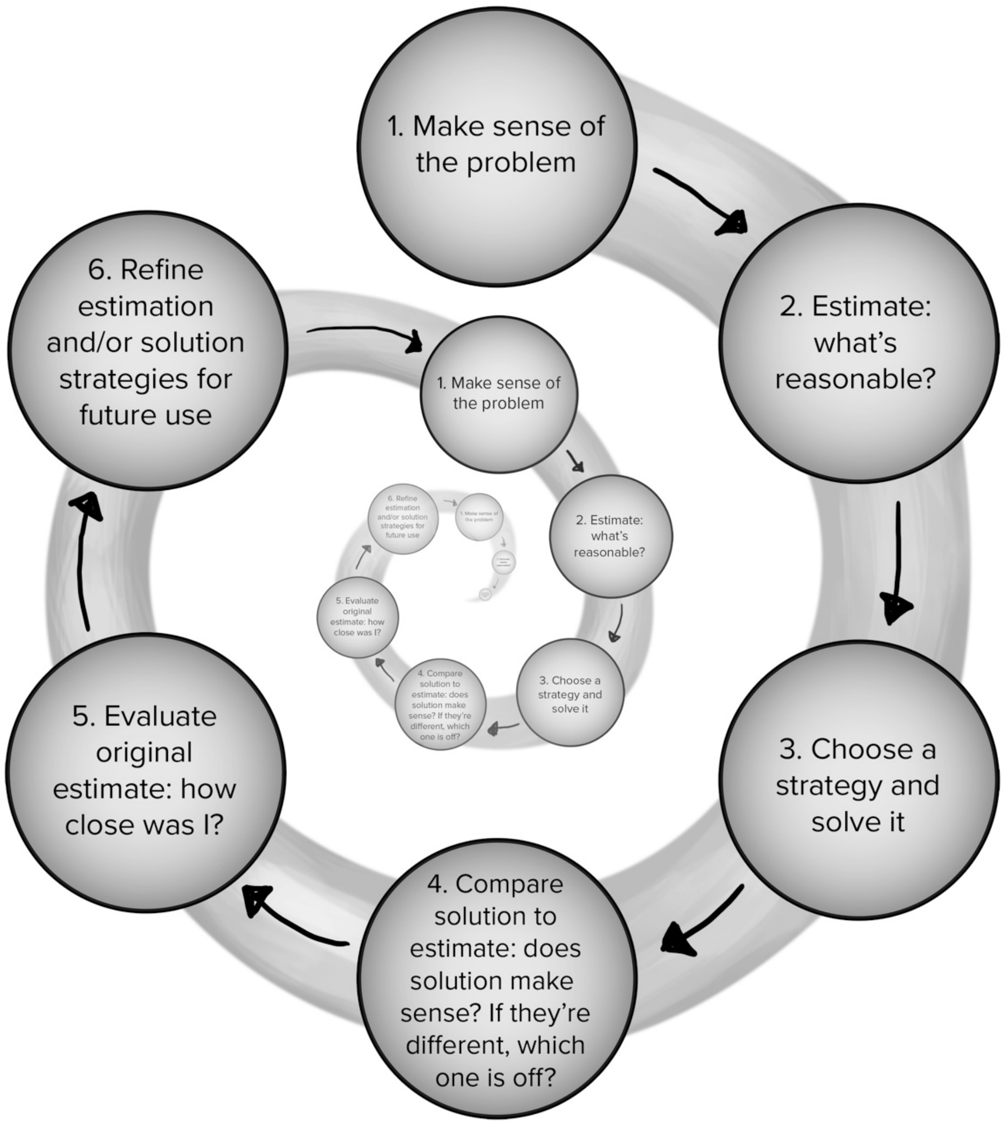
(The girls high-fived.)

Karni: We were close!

Alejandro and Leon

Alejandro estimated an angle to be 180 degrees and it turned out to be 120 degrees. He stopped and looked at all the angles with a furrowed brow. After a long time, his eyebrows shot up and he said: "Oh! I see what I did! I thought these" (picked up brown rhombuses) "were forty-five degrees. I pictured four of them. It was four of them, but they're not forty-five degrees! They're thirty degrees. That makes sense now." He resumed thinking quietly for another minute, then said, "Huh. They're smaller than I thought. Which one was forty-five degrees then?"

Alejandro and Leon found an angle they knew to be forty-five degrees and compared it to the thirty-degree angle. They stacked them on top of each other and carefully looked at the difference.



Zager, Tracy Johnston. 2017. *Becoming the Math Teacher You Wish You'd Had: Ideas and Strategies from Vibrant Classrooms.* Stenhouse Publishers: Portland, ME.

3,998 x 7

Jen: I wanted to show another example in which estimation—checking your work for reasonableness—can come to the rescue. (In this case, Jen rewrote a student's work on the whiteboard and shared the example anonymously.) Does anyone see anything up there . . . any red flags up there . . . that say, "That doesn't make sense! Oh my goodness, that doesn't make sense!" (Said in a comic, panicked voice, with arms waving.)

Eddie: That should be 21,000 instead of 2,100.

Jen: I agree, but why doesn't this make sense?

Annie: It's ten times smaller than it should be.

Jen: But how did you even know that it should have been ten times bigger? How does this one stick out and say, "That doesn't even make sense! That's unreasonable!" (Pointing to 3,000 × 7 = 2,100.)

Joseph: I think, because you took 3,000 × 7, and 2,100 is already *below* 3,000 before you start multiplying. You should think, 'That's the craziest answer ever!' It doesn't make sense that it's smaller than what you started with.

Mateo: And also, 900 × 7 is 6,300, which is bigger than 2,100, even though 900 is less than 3,000 and they're both being multiplied by 7.

Jen: Do you hear what they're saying? This number is smaller than this one, and you're multiplying both by 7, so how can this answer be bigger than this one? (Pointing to 900 and 3,000, then 2,100 and 6,300.) And if you did one group of 3,000, it's already bigger than 2,100, so that doesn't make sense.

The Cat Challenge

There are 7 girls on a bus.

Each girl has 7 backpacks.

In each backpack there are 7 big cats.

For every big cat, there are 7 little cats.

Question: How many legs are on the bus?

Maribel: "There are 7 girls on a bus. Each girl has 7 backpacks. In each backpack there are 7 big cats." So there were 49 backpacks, and each backpack has 7 cats, so that's 49 times 7.

Tania: So estimating, that's pretty close to 50 times 7, which is 350, right?

Janaya: What if we did 50 times 7 and then minus a 7, which is . . . 350 minus 7 . . . 343.

(Tania and Janaya wrote and thought. Maribel didn't write, but was clearly thinking.)

Maribel: We could do, like we could do 7 times 40 is 280. And 7 times 9 is 63. 280 plus 63 is . . .

Janaya: But we rounded to 50 and went back down.

Maribel: You did. It's easier for me to do it this way. So 280 plus 63 is 343. Is that what you found?

Tania: Yeah. We got the same thing two ways.

Janaya: Cool. That was a good way to check.

Maribel: So it equals 343. 343 *what*?

Tania: 343...

(Long pause)

Maribel: Let's reread the problem.

(They each reread the problem, and then thought quietly for about a minute.)

Janaya: Three hundred and forty-three cats. I think it's 343 cats.

Maribel: Oh my god. Each of the cats has 4 legs!

(Everybody laughed.)

Tania: Oh my god. That means 343 divided by 4 to figure out the legs.

Janaya: No, 343 times 4.

Maribel: Should we start over because we forgot we were thinking about legs?

Tania: Yes.

The same group, several minutes later:

Maribel: So 7 big cats. No! 7-times-7 big cats. We have to do 7 times 7 again.

Tania: I'm just so confused. Too many sevens!

Janaya: When we get the answer of how many cats, we just need to multiply by 4 to find the number of legs.

Maribel: But humans have 2 legs!

Janaya: We're not counting the humans.

Maribel: We're not? Why not?